

QUASI-POTENTIALS AND REGULARIZATION OF CURRENTS, AND APPLICATIONS

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ABSTRACT. Let Y be a compact Kähler manifold. We show that the weak regularization K_n of Dinh and Sibony for the diagonal Δ_Y (see Section 2 for more detail) is compatible with wedge product in the following sense:

If T is a positive dd^c -closed (p, p) current and θ is a smooth (q, q) form then there is a sequence of positive dd^c -closed $(p+q, p+q)$ currents S_n whose masses converge to 0 so that $-S_n \leq K_n(T \wedge \theta) - K_n(T) \wedge \theta \leq S_n$ for all n .

We also prove a result concerning the quasi-potentials of positive closed currents. We give two applications of these results. First, we prove a corresponding compatibility with wedge product for the pullback operator defined in our previous paper. Second, we define an intersection product for positive dd^c -closed currents. This intersection is symmetric and has a local nature.

1. INTRODUCTION

This paper continues the work of our previous paper [12] on pullback of currents. Here we prove a compatible property on pullingback of a current of the form $T \wedge \theta$, where T is a pseudo- dd^c -plurisubharmonic (p, p) current and θ is a smooth (q, q) form. We will also define an intersection product for positive dd^c -closed currents. This intersection is symmetric and has a local nature.

We will need the following two technical results. The first concerns the quasi-potential of a positive closed (p, p) current T on a compact Kähler manifold Y . It is known that (see Dinh and Sibony[8], Bost, Gillet and Soule[3]) there is a *DSH* $(p-1, p-1)$ current S and a closed smooth form α so that $T = \alpha + dd^c S$. (The definition of *DSH* currents, which was given by Dinh and Sibony [6], will be recalled in Section 3). Here S is a difference of two negative currents. When $p = 1$ or Y is a projective space, then we can choose S to be negative. However in general we can not choose S to be negative (see [3]). The following weaker conclusion is sufficient for the purpose of this paper

Lemma 1. *Let T be a positive closed (p, p) current on a compact Kähler manifold Y . Then there is a closed smooth (p, p) form α and a negative *DSH* $(p-1, p-1)$ current S so that*

$$T \leq \alpha + dd^c S.$$

Moreover, there is a constant $C > 0$ independent of T so that $\|\alpha\|_{L^\infty} \leq C\|T\|$ and $\|S\| \leq C\|T\|$. If T is strongly positive then we can choose S to be strongly negative.

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Here $||\cdot||_{L^\infty}$ is the maximum norm of a continuous form and $||\cdot||$ is the mass of a positive or negative current.

The second technical result concerns regularization of currents. If Y is a compact Kähler manifold, we let $\pi_1, \pi_2 : Y \times Y \rightarrow Y$ the projections. If K is a current on $Y \times Y$ and T a current on Y , we define $K(T) = (\pi_1)_*(K \wedge \pi_2^*(T))$, whenever the wedge product $K \wedge \pi_2^*(T)$ makes sense.

Lemma 2. *Let Y be a compact Kähler manifold. Let K_n be a weak regularization of the diagonal Δ_Y defined in [6] (see Section 2 for more detail). Let T be a DSH (p, p) current and let θ be a continuous (q, q) form on Y . Assume that there is a positive dd^c -closed current R so that $-R \leq T \leq R$. Then there are positive dd^c -closed $(p + q, p + q)$ currents R_n so that $\lim_{n \rightarrow \infty} ||R_n|| = 0$ and*

$$-R_n \leq K_n(T \wedge \theta) - K_n(T) \wedge \theta \leq R_n,$$

for all n .

If R is strongly positive or closed then we can choose R_n to be so.

Now we present some consequences of Lemmas 1 and 2. We discuss first the application to pullback of currents. Let X and Y be compact Kähler manifolds and let $f : X \rightarrow Y$ be a dominant meromorphic map. In [12] we defined a pullback operator f^\sharp for currents on Y as follows. Let $s \geq 0$ be an integer. Then a good approximation scheme by C^s forms is an approximation for all DSH currents by C^s forms and satisfies a list of requirements (See Definition 4 in Section 3. Note that the definition of good approximation schemes here is stronger than that in [12] because here we require it to satisfy in addition the conclusions of Lemma 2). Because Y is compact, if T is a current on Y then it is of a finite order s_0 . We say that $f^\sharp(T) = S$ is well-defined if there is a number $s \geq s_0$ such that for any good approximation by C^{s+2} forms \mathcal{K}_n then for any smooth form α on X we have

$$\lim_{n \rightarrow \infty} \int_Y T \wedge \mathcal{K}_n(f_*(\alpha)) = \int_X S \wedge \alpha.$$

Let Γ_f be the graph of f . Let $\pi_X, \pi_Y : X \times Y \rightarrow X, Y$ be the projections. A current τ is called pseudo- dd^c -plurisubharmonic if there is a smooth form γ so that $dd^c \tau \geq -\gamma$. We have the following result

Theorem 1. *Let T be a DSH (p, p) current and let θ be a smooth (q, q) form on Y . Assume that there is a positive pseudo- dd^c -plurisubharmonic current τ so that $-\tau \leq T \leq \tau$.*

a) *If f is holomorphic and $f^\sharp(T)$ is well-defined, then $f^\sharp(T \wedge \theta)$ is well-defined. Moreover, $f^\sharp(T \wedge \theta) = f^\sharp(T) \wedge f^*(\theta)$.*

b) *More general, assume that there is a number $s \geq 0$ and a (p, p) current $(\pi_Y|_{\Gamma_f})^\sharp(T)$ on $X \times Y$ such that for any good approximation by C^{s+2} forms \mathcal{K}_n then*

$$\lim_{n \rightarrow \infty} \pi_Y^*(\mathcal{K}_n(T)) \wedge [\Gamma_f] = (\pi_Y|_{\Gamma_f})^\sharp(T).$$

Then $f^\sharp(T \wedge \theta)$ is well-defined, and moreover $f^\sharp(T \wedge \theta) = (\pi_X)_((\pi_Y|_{\Gamma_f})^\sharp(T) \wedge \pi_Y^*(\theta))$.*

Roughly speaking, the result b) of Theorem 1 says that under some natural conditions if we can pullback T then we can do it locally. To illustrate the use of Theorem 1 we will show in Proposition 1 in Section 3 the following result: If T is a

(p, p) (non-smooth) form whose coefficients are bounded by a quasi-PSH function then T can be pulled back by meromorphic maps. Moreover the resulting current is the same as that defined by Dinh and Sibony (see Proposition 4.2 in [7])

Some special cases of Theorem 1 and Proposition 1 have been considered in the literature. Diller [5] defined for a rational selfmap of \mathbb{P}^2 the pullback of a current of the form ψT where ψ is a smooth function and T is a positive closed $(1, 1)$ current. Russakovskii and Shiffman [11] defined pullback by a holomorphic map for currents of the forms $\varphi \Phi$ (where φ is a quasi-plurisubharmonic function and Φ is a smooth form) and $[D] \wedge \Phi$ (where D is a divisor and Φ is a smooth form).

We can also apply Theorem 1 to other situations. Let $f : X \rightarrow Y$ be a dominant meromorphic map between compact Kähler manifolds. In [12], we showed that if T is a positive dd^c -closed $(1, 1)$ current then $f^\#(T)$ is well-defined (the resulting current coincides with the definitions given by Alessandrini and Bassanelli [1] and Dinh and Sibony [7]), and therefore $f^\#(T \wedge \theta)$ is well-defined for any smooth (q, q) form θ . Likewise, if V is an irreducible analytic variety of codimension p so that $\pi_Y^{-1}(V) \cap \Gamma_f$ has codimension $\geq p$ then $(\pi_Y|_{\Gamma_f})^\#[V]$ is well-defined, and therefore $f^\#([V] \wedge \theta)$ is well-defined for smooth (q, q) forms θ .

Using super-potential theory, Dinh and Sibony [9] defined a satisfying intersection theory for positive closed currents on a projective space. In the below we give a definition for intersection product of currents on a general compact Kähler manifold and discuss some of its properties.

Definition 2. *Let Y be a compact Kähler manifold. Let T_1 be a DSH current and let T_2 be any current on Y . Let s_0 be the order of T_2 . We say that $T_1 \wedge T_2$ is well-defined if there is $s \geq s_0$ and a current S so that for any good approximation scheme by C^{s+2} forms \mathcal{K}_n then $\lim_{n \rightarrow \infty} \mathcal{K}_n(T_1) \wedge T_2 = S$. Then we write $T_1 \wedge T_2 = S$.*

This definition has the following properties

Theorem 3. *Let T_1 and T_2 be positive dd^c -closed currents. Assume that $T_1 \wedge T_2$ is well-defined. Let θ be a smooth (q, q) form.*

- a) $\theta \wedge T_2$ and $T_2 \wedge \theta$ are well-defined and are the same as the usual definition.*
- b) $T_2 \wedge T_1$ is also well-defined. Moreover, $T_1 \wedge T_2 = T_2 \wedge T_1$.*
- c) $T_1 \wedge (\theta \wedge T_2)$ is also well-defined. Moreover $T_1 \wedge (\theta \wedge T_2) = (T_1 \wedge T_2) \wedge \theta$.*

Theorem 3 b) means that the intersection is symmetric, and Theorem 3 c) means that the intersection can be computed locally. For intersection of varieties we have the expected result

Lemma 3. *Let V_1 and V_2 be irreducible subvarieties of codimensions p and q of Y . Assume that any component of $V_1 \cap V_2$ has codimension $p + q$. Then $[V_1] \wedge [V_2]$ is well-defined. Here $[V_1]$ and $[V_2]$ are the currents of integration on V_1 and V_2 .*

The rest of this paper is organized as follows. In Section 2 we recall the construction of weak regularization for the diagonal and prove Lemmas 1 and 2. In Section 3 we prove the other results.

2. PROOFS OF LEMMAS 1 AND 2

Let Y be a compact Kähler manifold of dimension k . Let $\pi_1, \pi_2 : Y \times Y \rightarrow Y$ be the two projections, and let $\Delta_Y \subset Y \times Y$ be the diagonal. Let ω_Y be a Kähler $(1, 1)$ form on Y .

For any p , we define $DSH^p(Y)$ (see [6]) to be the space of (p, p) currents $T = T_1 - T_2$, where T_i are positive currents, such that $dd^c T_i = \Omega_i^+ - \Omega_i^-$ with Ω_i^\pm positive closed. Observe that $\|\Omega_i^+\| = \|\Omega_i^-\|$ since they are cohomologous to each other because $dd^c(T_i)$ is an exact current. Define the DSH -norm of T as

$$\|T\|_{DSH} := \min\{\|T_1\| + \|T_2\| + \|\Omega_1^+\| + \|\Omega_2^+\|, T_i, \Omega_i, \text{ as above}\}.$$

Using compactness of positive currents, it can be seen that we can find T_i, Ω_i^\pm which realize $\|T\|_{DSH}$, hence the minimum on the RHS of the definition of DSH norm. We say that $T_n \rightharpoonup T$ in $DSH^p(Y)$ if T_n weakly converges to T and $\|T_n\|_{DSH}$ is bounded.

Recall that a function φ is quasi-PSH if it is upper semi-continuous, belongs to L^1 , and $dd^c(\varphi) = T - \theta$, where T is a positive closed $(1, 1)$ current and θ is a closed smooth $(1, 1)$ form. We also call φ a θ -plurisubharmonic function.

Remark 1. *The following consideration from [3] and [8] is used in both proof of Lemam 1 and the construction of the kernels K_n in Lemma 2. Let $k = \text{dimension of } Y$. Let $\pi : \tilde{Y} \times Y \rightarrow Y \times Y$ be the blowup of $Y \times Y$ at Δ_Y . Let $\tilde{\Delta}_Y = \pi^{-1}(\Delta_Y)$ be the exceptional divisor. Then there is a closed smooth $(1, 1)$ form γ and a negative quasi-plurisubharmonic function φ so that $dd^c \varphi = [\tilde{\Delta}_Y] - \gamma$. We choose a strictly positive closed smooth $(k-1, k-1)$ form η so that $\pi_*([\tilde{\Delta}_Y] \wedge \eta) = [\Delta_Y]$.*

Next we give the proof of Lemma 1.

Proof. (Of Lemma 1) Notations are as in Remark 1. Define $H = \pi_*(\varphi\eta)$. Then H is a negative $(k-1, k-1)$ current on $Y \times Y$.

We write $\gamma = \gamma^+ - \gamma^-$ for strictly positive closed smooth $(1, 1)$ forms γ^\pm . If we define $\Phi^\pm = \pi_*(\gamma^\pm \wedge \eta)$ then Φ^\pm are positive closed (k, k) currents with L^1 coefficients. In fact (see [6]) Φ^\pm are smooth away from the diagonal Δ_Y , and the singularities of $\Phi^\pm(y_1, y_2)$ and their derivatives are bounded by $|y_1 - y_2|^{-(2k-2)}$ and $|y_1 - y_2|^{-(2k-1)}$. Moreover

$$dd^c H = \pi_*(dd^c \varphi \wedge \eta) = \pi_*([\tilde{\Delta}_Y] \wedge \eta - (\gamma^+ - \gamma^-) \wedge \eta) = [\Delta_Y] - (\Phi^+ - \Phi^-).$$

Consider $S_1 = (\pi_1)_*(H \wedge \pi_2^*(T))$ and $R_1^\pm = (\pi_1)_*(\Phi^\pm \wedge T)$. Then S_1 is a negative current, and R_1^\pm are positive closed currents. Moreover

$$dd^c S_1 = (\pi_1)_*(dd^c H \wedge \pi_2^*(T)) = T - R_1^+ + R_1^-.$$

Therefore $T \leq R_1^+ + dd^c S_1$. Moreover R_1^+ is a current with L^1 coefficients, and there is a constant $C_1 > 0$ independent of T so that $\|S_1\|, \|R_1\|_{L^1} \leq C_1 \|T\|$ (see e.g. Lemma 2.1 in [6]).

If we apply this process for R_1^+ instead of T we find a positive closed current R_2^+ with coefficients in $L^{1+1/(2k+2)}$ and a negative current S_2 so that $R_1^+ \leq R_2^+ + dd^c S_2$. Moreover

$$\|R_2^+\|_{L^{1+1/(2k+2)}}, \|S_2\| \leq C_2 \|R_1^+\|_{L^1} \leq C_1 C_2 \|T\|$$

for some constant $C_2 > 0$ independent of T . After iterating this process a finite number of times we find a continuous form R and a negative current S so that $T \leq R + dd^c S$. Moreover, $\|R\|_{L^\infty}, \|S\| \leq C \|T\|$ for some constant $C > 0$ independent of T . Since we can bound R by ω_Y^p upto a multiple constant of size $\|R\|_{L^\infty}$, we are done. \square

Next we recall the construction of the kernels K_n from Section 3 in [6]. Notations are as in Remark 1. Observe that φ is smooth out of $[\widetilde{\Delta_Y}]$, and $\varphi^{-1}(-\infty) = \widetilde{\Delta_Y}$. Let $\chi : \mathbb{R} \cup \{-\infty\} \rightarrow \mathbb{R}$ be a smooth increasing convex function such that $\chi(x) = 0$ on $[-\infty, -1]$, $\chi(x) = x$ on $[1, +\infty]$, and $0 \leq \chi' \leq 1$. Define $\chi_n(x) = \chi(x+n) - n$, and $\varphi_n = \chi_n \circ \varphi$. The functions φ_n are smooth decreasing to φ , and $dd^c \varphi_n \geq -\Theta$ for every n , where Θ is a strictly positive closed smooth $(1, 1)$ form so that $\Theta - \gamma$ is strictly positive. Then we define $\Theta_n^+ = dd^c \varphi_n + \Theta$ and $\Theta_n^- = \Theta^- = \Theta - \gamma$. Finally $K_n^\pm = \pi_*(\Theta_n^\pm \wedge \eta)$, and $K_n = K_n^+ - K_n^-$.

Proof. (Of Lemma 2)

Let us define $H_n = K_n(T \wedge \theta) - K_n(T) \wedge \theta$. Since T and θ may not be either positive or dd^c -closed, a priori H_n is neither. However, we will show that there are positive dd^c -closed currents R_n such that $\lim_{n \rightarrow \infty} \|R_n\| = 0$ and $-R_n \leq H_n \leq R_n$.

By definition we have

$$H_n(y) = \int_{z \in Y} K_n(y, z) \wedge (\theta(z) - \theta(y)) \wedge T(z).$$

Fix a number $\delta > 0$. Then by the construction of K_n , there is an integer n_δ so that if $n \geq n_\delta$ and $|y - z| \geq \delta$ then $K_n(y, z) = 0$. Thus

$$H_n(y) = \int_{z \in Y, |z-y| < \delta} (K_n^+(y, z) - K_n^-(y, z)) \wedge (\theta(z) - \theta(y)) \wedge T(z).$$

We define $h(\delta) = \max_{y, z \in Y: |y-z| \leq \delta} |\theta(y) - \theta(z)|$. Because θ is a continuous form, we have $\lim_{\delta \rightarrow 0} h(\delta) = 0$. Moreover, since $Y \times Y$ is compact, there is a constant $C > 0$ independent of θ and δ so that

$$-h(\delta)C(\omega_Y(y) + \omega_Y(z))^q \leq \theta(z) - \theta(y) \leq h(\delta)C(\omega_Y(y) + \omega_Y(z))^q$$

for all $\delta \leq 1$ and for all $|y - z| \leq \delta$. Since $K_n^\pm(y, z)$ are strongly positive closed and $-R \leq T \leq R$, it follows that

$$\begin{aligned} H_n(y) &= \int_{z \in Y, |z-y| < \delta} (K_n^+(y, z) - K_n^-(y, z)) \wedge (\theta(z) - \theta(y)) \wedge T(z) \\ &\leq h(\delta)C \int_{z \in Y, |z-y| < \delta} (K_n^+(y, z) + K_n^-(y, z)) \wedge (\omega_Y(y) + \omega_Y(z))^q \wedge R(z) \\ &\leq h(\delta)C \int_{z \in Y} (K_n^+(y, z) + K_n^-(y, z)) \wedge (\omega_Y(y) + \omega_Y(z))^q \wedge R(z). \end{aligned}$$

Thus $H_n(y) \leq R_n(y)$ where

$$R_n(y) = h(\delta)C \int_{z \in Y} (K_n^+(y, z) + K_n^-(y, z)) \wedge (\omega_Y(y) + \omega_Y(z))^q \wedge R(z),$$

for $n_\delta \leq n < n_{\delta/2}$. Similarly we have $H_n(y) \geq -R_n(y)$. It can be checked that $R_n(y)$ is positive dd^c -closed. Moreover, there is a constant $C_1 > 0$ independent of n, δ, R and θ so that

$$(2.1) \quad \|R_n\| \leq h(\delta)C_1\|R\|,$$

for $n \geq n_\delta$. This shows that $\|R_n\| \rightarrow 0$ as $n \rightarrow \infty$. \square

Remark 2. By the estimate (2.1) and by iterating we obtain the following result: Let T , R and θ be as in Lemma 2. Then there are positive dd^c -closed $(p+q, p+q)$ currents R_{n_1, n_2, \dots, n_l} so that

$$-R_{n_1, n_2, \dots, n_l} \leq K_{n_1} \circ K_{n_2} \circ \dots \circ K_{n_l}(T \wedge \theta) - K_{n_1} \circ K_{n_2} \circ \dots \circ K_{n_l}(T) \wedge \theta \leq R_{n_1, n_2, \dots, n_l},$$

and

$$\lim_{n_1, n_2, \dots, n_l \rightarrow \infty} \|R_{n_1, n_2, \dots, n_l}\| = 0.$$

We give the proof of this claim for example when $l = 2$. We will write the R_n in Lemma 2 by $R_n(R)$ to emphasize its dependence on R . Writing

$$\begin{aligned} & K_{n_1} \circ K_{n_2}(T \wedge \theta) - K_{n_1} \circ K_{n_2}(T) \wedge \theta \\ &= [K_{n_1}(K_{n_2}(T \wedge \theta) - K_{n_2}(T) \wedge \theta)] + [K_{n_1}(K_{n_2}(T) \wedge \theta) - K_{n_1}(K_{n_2}(T)) \wedge \theta], \end{aligned}$$

and choosing

$$R_{n_1, n_2} = K_{n_1}^+(R_{n_2}(R)) + K_{n_1}^-(R_{n_2}(R)) + R_{n_1}(K_{n_2}^+(R)) + R_{n_1}(K_{n_2}^-(R)),$$

we see that

$$-R_{n_1, n_2} \leq K_{n_1} \circ K_{n_2}(T \wedge \theta) - K_{n_1} \circ K_{n_2}(T) \wedge \theta \leq R_{n_1, n_2}.$$

That R_{n_1, n_2} are positive dd^c -closed follows from the properties of the kernels K_n . It remains to bound the masses of R_{n_1, n_2} . By (2.1) we have

$$\begin{aligned} \|R_{n_1, n_2}\| &\leq C_1(\|R_{n_2}(R)\| + \|R_{n_1}(K_{n_2}^+(R))\| + \|R_{n_2}(K_{n_2}^-(R))\|) \\ &\leq C_2 h(\delta)(\|R\| + \|K_{n_2}^+(R)\| + \|K_{n_2}^-(R)\|) \\ &\leq C_3 h(\delta)\|R\|, \end{aligned}$$

for constants C_1, C_2, C_3 and for all $n_1, n_2 \geq n_\delta$, here n_δ is the constant in the proof of Lemma 2.

3. PROOFS OF THE CONSEQUENCES

We first give the definition of a good approximation scheme by C^s forms for DSH currents.

Definition 4. Let Y be a compact Kahler manifold. Let $s \geq 0$ be an integer. We define a good approximation scheme by C^s forms for DSH currents on Y to be an assignment that for a DSH current T gives two sequences $\mathcal{K}_n^\pm(T)$ (here $n = 1, 2, \dots$) where $\mathcal{K}_n^\pm(T)$ are C^s forms of the same bidegrees as T , so that $\mathcal{K}_n(T) = \mathcal{K}_n^+(T) - \mathcal{K}_n^-(T)$ weakly converges to T , and moreover the following properties are satisfied:

- 1) Boundedness: The DSH norms of $\mathcal{K}_n^\pm(T)$ are uniformly bounded.
- 2) Positivity: If T is positive then $\mathcal{K}_n^\pm(T)$ are positive, and $\|\mathcal{K}_n^\pm(T)\|$ is uniformly bounded with respect to n .
- 3) Closedness: If T is positive closed then $\mathcal{K}_n^\pm(T)$ are positive closed.
- 4) Continuity: If $U \subset Y$ is an open set so that $T|_U$ is a continuous form then $\mathcal{K}_n^\pm(T)$ converges locally uniformly on U .
- 5) Additivity: If T_1 and T_2 are two DSH^p currents, then $\mathcal{K}_n^\pm(T_1 + T_2) = \mathcal{K}_n^\pm(T_1) + \mathcal{K}_n^\pm(T_2)$.

6) *Commutativity:* If T and S are DSH currents with complements bidegrees then

$$\lim_{n \rightarrow \infty} \left[\int_Y \mathcal{K}_n(T) \wedge S - \int_Y T \wedge \mathcal{K}_n(S) \right] = 0.$$

7) *Compatibility with the differentials:* $dd^c \mathcal{K}_n^\pm(T) = \mathcal{K}_n^\pm(dd^c T)$.

8) *Condition on support:* The support of $\mathcal{K}_n(T)$ converges to the support of T . By this we mean that if U is an open neighborhood of $\text{supp}(T)$, then there is n_0 so that when $n \geq n_0$ then $\text{supp}(\mathcal{K}_n(T))$ is contained in U . Moreover, the number n_0 can be chosen so that it depends only on $\text{supp}(T)$ and U but not on the current T .

9) *Compatibility with wedge product:* Let T be a DSH (p, p) current and let θ be a continuous (q, q) form on Y . Assume that there is a positive dd^c -closed current R so that $-R \leq T \leq R$. Then there are positive dd^c -closed $(p+q, p+q)$ currents R_n so that $\lim_{n \rightarrow \infty} \|R_n\| = 0$ and

$$-R_n \leq \mathcal{K}_n(T \wedge \theta) - \mathcal{K}_n(T) \wedge \theta \leq R_n,$$

for all n .

If R is strongly positive or closed then we can choose R_n to be so.

Let K_n be the weak regularization for the diagonal Δ_Y as in Section 2. Let l be a large integer dependent on s , and let $(m_1)_n, \dots, (m_l)_n$ be sequences of positive integers satisfying $(m_i)_n = (m_{l+1-i})_n$ and $\lim_{n \rightarrow \infty} (m_i)_n = \infty$ for any $1 \leq i \leq l$. In [12] we showed that if we choose $\mathcal{K}_n = K_{(m_1)_n} \circ K_{(m_2)_n} \circ \dots \circ K_{(m_l)_n}$ then it satisfies conditions 1)-8). Remark 2 shows that it also satisfies condition 9).

Note that by condition 6), if T is a DSH current then $f^\sharp(T)$ is well-defined iff there is a number $s \geq 0$ and a current S so that for any good approximation scheme by C^{s+2} forms \mathcal{K}_n then $\lim_{n \rightarrow \infty} f^*(\mathcal{K}_n(T)) = S$.

Proof. (Of Theorem 1)

a) We let $s \geq 0$ be a number so that for any good approximation scheme by C^{s+2} forms \mathcal{K}_n and for any smooth form α on X then

$$\int_X f^\sharp(T) \wedge \alpha = \lim_{n \rightarrow \infty} \int_Y T \wedge \mathcal{K}_n(f_*(\alpha)).$$

Then for the proof of a) it suffices to show that for any smooth form β on X then

$$\lim_{n \rightarrow \infty} \int_Y T \wedge \theta \wedge \mathcal{K}_n(f_*(\beta)) = \int_X f^\sharp(T) \wedge f^*(\theta) \wedge \beta.$$

If we can show

$$(3.1) \quad \lim_{n \rightarrow \infty} \int_Y T \wedge (\theta \wedge \mathcal{K}_n(f_*(\beta)) - \mathcal{K}_n(\theta \wedge f_*(\beta))) = 0$$

then we are done, since we have $\theta \wedge f_*(\beta) = f_*(f^*(\theta) \wedge \beta)$ because f is holomorphic, and hence

$$\lim_{n \rightarrow \infty} \int_Y T \wedge \mathcal{K}_n(\theta \wedge f_*(\beta)) = \lim_{n \rightarrow \infty} \int_Y T \wedge \mathcal{K}_n(f_*(f^*(\theta) \wedge \beta)) = \int_Y f^\sharp(T) \wedge (f^*(\theta) \wedge \beta).$$

Now we proceed to proving (3.1). For a fixed n we have

$$\begin{aligned} & \int_Y T \wedge (\theta \wedge \mathcal{K}_n(f_*(\beta)) - \mathcal{K}_n(\theta \wedge f_*(\beta))) \\ &= \lim_{m \rightarrow \infty} \int_Y \mathcal{K}_m(T) \wedge (\theta \wedge \mathcal{K}_n(f_*(\beta)) - \mathcal{K}_n(\theta \wedge f_*(\beta))). \end{aligned}$$

The advantage of this is that $\mathcal{K}_m(T)$ are continuous forms, hence if we have bounds of $\theta \wedge \mathcal{K}_n(f_*(\beta)) - \mathcal{K}_n(\theta \wedge f_*(\beta))$ by currents of order zero we can use them in the integral and then take limit when $m \rightarrow \infty$.

Because $f_*(\beta)$ is bound by a multiple of $f_*(\omega_X^{\dim(X)-p-q})$ and the latter is strongly positive closed, by condition 9) of Definition 4 there are strongly positive closed currents R_n with $\|R_n\| \rightarrow 0$ and

$$-R_n \leq \theta \wedge \mathcal{K}_n(f_*(\beta)) - \mathcal{K}_n(\theta \wedge f_*(\beta)) \leq R_n,$$

for all n . Since $-\tau \leq T \leq \tau$, we have $-(\mathcal{K}_m^+(\tau) + \mathcal{K}_m^-(\tau)) \leq \mathcal{K}_m(T) \leq \mathcal{K}_m^+(\tau) + \mathcal{K}_m^-(\tau)$. Since $\mathcal{K}_m^+(\tau) + \mathcal{K}_m^-(\tau)$ are positive C^2 forms, from the above estimates we obtain

$$\begin{aligned} - \int_Y (\mathcal{K}_m^+(\tau) + \mathcal{K}_m^-(\tau)) \wedge R_n &\leq \int_Y \mathcal{K}_m(T) \wedge (\theta \wedge \mathcal{K}_n(f_*(\beta)) - \mathcal{K}_n(\theta \wedge f_*(\beta))) \\ &\leq \int_Y (\mathcal{K}_m^+(\tau) + \mathcal{K}_m^-(\tau)) \wedge R_n. \end{aligned}$$

Hence (3.1) follows if we can show that

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_Y (\mathcal{K}_m^+(\tau) + \mathcal{K}_m^-(\tau)) \wedge R_n = 0.$$

By Lemma 1, there are a smooth closed form α_n and a strongly negative current S_n for which $R_n \leq \alpha_n + dd^c S_n$ and $\|\alpha_n\|_{L^\infty}, \|S_n\| \rightarrow 0$. Therefore

$$\begin{aligned} 0 &\leq \int_Y (\mathcal{K}_m^+(\tau) + \mathcal{K}_m^-(\tau)) \wedge R_n \\ &\leq \int_Y (\mathcal{K}_m^+(\tau) + \mathcal{K}_m^-(\tau)) \wedge \alpha_n + \int_Y (\mathcal{K}_m^+(\tau) + \mathcal{K}_m^-(\tau)) \wedge dd^c S_n. \end{aligned}$$

Since the currents $\mathcal{K}_m^\pm(\tau)$ are positive whose masses are uniformly bounded, it follows from $\|\alpha_n\|_{L^\infty} \rightarrow 0$ that

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_Y (\mathcal{K}_m^+(\tau) + \mathcal{K}_m^-(\tau)) \wedge \alpha_n = 0.$$

Now we estimate the other term. We have

$$\int_Y (\mathcal{K}_m^+(\tau) + \mathcal{K}_m^-(\tau)) \wedge dd^c S_n = \int_Y (\mathcal{K}_m^+(dd^c \tau) + \mathcal{K}_m^-(dd^c \tau)) \wedge S_n.$$

Because S_n is strongly negative and $dd^c \tau \geq -\gamma$, the last integral can be bound from above by

$$\int_Y (\mathcal{K}_m^+(dd^c \tau) + \mathcal{K}_m^-(dd^c \tau)) \wedge S_n \leq \int_Y (\mathcal{K}_m^+(-\gamma) + \mathcal{K}_m^-(-\gamma)) \wedge S_n.$$

Since γ is smooth, by condition 4) of Definition 4 and the fact that $\|S_n\| \rightarrow 0$, we obtain

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_Y (\mathcal{K}_m^+(-\gamma) + \mathcal{K}_m^-(-\gamma)) \wedge S_n = 0.$$

Thus, whatever the limit of

$$\int_Y (\mathcal{K}_m^+(\tau) + \mathcal{K}_m^-(\tau)) \wedge dd^c S_n$$

is, it is non-positive. The proof of (3.1) and hence of a) is finished.

b) The proof of b) is similar to that of a). \square

Now we give an application to pulling back of (non-smooth) forms whose coefficients are bounded by a quasi-PSH function.

Proposition 1. *Let T be a (p, p) form whose coefficients are bounded by a quasi-PSH function φ . Then $f^\sharp(T)$ is well-defined.*

Proof. By desingularizing the graph Γ_f if needed and using Theorem 4 in [12], we can assume without loss of generality that f is holomorphic. By subtracting a constant from φ if needed, we can assume that $\varphi \leq 0$. By using partition of unity, we reduce the problem to the case where $T = \psi\theta$ where ψ is a function with $0 \geq \psi \geq \varphi$ and θ is a smooth form. By Theorem 1 a), for a proof of Proposition 1 it suffices to show that $f^\sharp(\psi)$ is well-defined. To this end we will show the existence of a current S so that for any smooth form α and any good approximation scheme by C^2 forms \mathcal{K}_n then

$$(3.2) \quad \lim_{n \rightarrow \infty} \int_Y \psi \wedge \mathcal{K}_n(f_*(\alpha)) = \int_X S \wedge \alpha.$$

We define linear functionals S_n and S_n^\pm on top forms on X by the formulas

$$\begin{aligned} \langle S_n, \alpha \rangle &= \int_Y \psi \wedge \mathcal{K}_n(f_*(\alpha)), \\ \langle S_n^\pm, \alpha \rangle &= \int_Y \psi \wedge \mathcal{K}_n^\pm(f_*(\alpha)). \end{aligned}$$

Then $S_n = S_n^+ - S_n^-$, and it can be checked that S_n^\pm are negative $(0, 0)$ currents, and hence S_n is a current of order 0. Moreover, if α is a positive smooth measure then

$$\begin{aligned} 0 \geq \langle S_n^\pm, \alpha \rangle &= \int_Y \psi \wedge \mathcal{K}_n^\pm(f_*(\alpha)) \\ &\geq \int_Y \varphi \wedge \mathcal{K}_n^\pm(f_*(\alpha)) \\ &= \int_X f^*(\mathcal{K}_n^\pm(\varphi)) \wedge \alpha. \end{aligned}$$

Thus $0 \geq S_n^\pm \geq f^*(\mathcal{K}_n^\pm(\varphi))$ for all n .

Let us write $dd^c(\varphi) = T - \theta$ where T is a positive closed $(1, 1)$ current, and θ is a smooth closed $(1, 1)$ form. By property 4) of Definition 4, there is a strictly positive closed smooth $(1, 1)$ form Θ so that $\Theta \geq \mathcal{K}_n^\pm(\theta)$ for any n . Then $f^*(\mathcal{K}_n^\pm(\varphi))$ are negative C^2 forms so that

$$\begin{aligned} dd^c f^*(\mathcal{K}_n^\pm(\varphi)) &= f^*(\mathcal{K}_n^\pm(dd^c \varphi)) = f^*(\mathcal{K}_n^\pm(T - \theta)) \\ &\geq f^*(\mathcal{K}_n^\pm(-\theta)) \geq -f^*(\Theta) \end{aligned}$$

for any n , i.e they are negative $f^*(\Theta)$ -plurisubharmonic functions. Moreover the sequence of currents $f^*(\mathcal{K}_n^\pm(\varphi))$ has uniformly bounded mass (see the proof of Theorem 6 in [12]). Therefore, by the compactness of this class of functions (see

Chapter 1 in [4]), after passing to a subsequence if needed, we can assume that $f^*(\mathcal{K}_n^\pm(\varphi))$ converges in L^1 to negative functions denoted by $f^*(\varphi^\pm)$. Let S^\pm be any cluster points of S_n^\pm . Then $0 \geq S^\pm \geq f^*(\varphi^\pm)$, which shows that any cluster point $S = S^+ - S^-$ of S_n has no mass on sets of Lebesgue measure zero. Hence to show that S is uniquely defined, it suffices to show that S is uniquely defined outside a proper analytic subset of Y .

Let E be a proper analytic subset of Y so that $f : X - f^{-1}(E) \rightarrow Y - E$ is a holomorphic submersion. If α is a smooth measure whose support is compactly contained in $X - f^{-1}(E)$ then $f_*(\alpha)$ is a smooth measure on Y . Hence by condition 4) of Definition 4, $\mathcal{K}_n(f_*(\alpha))$ uniformly converges to the smooth measure $f_*(\alpha)$. Then it follows from the definition of S that

$$\langle S, \alpha \rangle = \int_Y \psi \wedge f_*(\alpha).$$

Hence S is uniquely defined on $X - E$, and thus it is uniquely defined on the whole X , as wanted. \square

Finally, we consider the intersection of currents.

Proof. (Of Theorem 3)

Proof of a): Let \mathcal{K}_n be a good approximation scheme by C^2 forms. Then $\mathcal{K}_n(\theta)$ uniformly converges to θ , and hence $\mathcal{K}_n(\theta) \wedge T_2$ converges to the usual intersection $\theta \wedge T_2$.

Let α be a smooth form. Then by conditions 9), 6) and 4) of Definition 4, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_Y \mathcal{K}_n(T_2) \wedge \theta \wedge \alpha &= \lim_{n \rightarrow \infty} \int_Y \mathcal{K}_n(T_2 \wedge \theta) \wedge \alpha \\ &= \lim_{n \rightarrow \infty} \int_Y T_2 \wedge \theta \wedge \mathcal{K}_n(\alpha) \\ &= \int_Y T_2 \wedge \theta \wedge \alpha. \end{aligned}$$

The proofs of b) and c) are similar. \square

Proof. (Of Lemma 3) Let θ be a smooth (p, p) form having the same cohomology class as that of $[V_1]$. Then by Proposition 2.1 in [8], there are positive $(p-1, p-1)$ currents R^\pm so that $[V_1] - \theta = dd^c(R^+ - R^-)$. Moreover, R^\pm are *DSH* and we can choose so that R^\pm are continuous outside V_1 . To prove Lemma 3, it suffices to show that there is a current S so that for any good approximation scheme by C^2 forms \mathcal{K}_n then

$$\lim_{n \rightarrow \infty} \mathcal{K}_n(R^+ - R^-) \wedge [V_2] = S.$$

The sequence $\mathcal{K}_n^\pm(R^\pm) \wedge [V_2]$ converges on $Y - V_1 \cap V_2$. In fact, outside of V_2 then $\mathcal{K}_n^\pm(R^\pm) \wedge [V_2] = 0$, and outside of V_1 then $\mathcal{K}_n^\pm(R^\pm)$ converges locally uniformly (by condition 4) of Definition 4) to a continuous form and hence $\mathcal{K}_n^\pm(R^\pm) \wedge [V_2]$ converges. Then by an argument as in the proof of Theorem 6 in [12] using the Federer-type support theorem in Bassanelli [2], we are done. \square

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